Exchange economy with cobb-douglas utility

Consider an exchange economy with two consumers (A, B) and two goods (x, y) in which the utility functions are:

$$U_A = x_A y_A, \quad U_B = x_B y_B.$$

The initial endowments of goods are:

$$\bar{x}_A = 90, \quad \bar{x}_B = 30$$
$$\bar{y}_A = 35, \quad \bar{y}_B = 25.$$

Let P_x and P_y be the prices of goods x and y respectively. Obtain the Walrasian equilibrium allocation (use the normalization $P_y = 1$).

Solution

The supply of good x is $\bar{x}_A + \bar{x}_B = 120$.

The supply of good y is $\bar{y}_A + \bar{y}_B = 60$.

Equilibrium conditions:

$$x_A^* + x_B^* = 120 = \bar{x}_A + \bar{x}_B$$

$$y_A^* + y_B^* = 60 = \bar{y}_A + \bar{y}_B$$

Normalization: $P_y = 1$

Budget constraints:

$$\text{Consumer A: } P_x x_A + y_A = 90P_x + 35$$

$$\text{Consumer B: } P_x x_B + y_B = 30P_x + 25$$

Consumer A's problem:

$$\max_{x_A, y_A} x_A y_A \quad \text{s.t.} \quad P_x x_A + y_A = 90P_x + 35$$

$$L = x_A y_A + \lambda(90P_x + 35 - P_x x_A - y_A)$$

First-order conditions:

$$\frac{\partial L}{\partial x_A} = 0 \implies y_A - \lambda P_x = 0 \implies \lambda = y_A \frac{1}{P_x}$$

$$\frac{\partial L}{\partial y_A} = 0 \implies x_A - \lambda = 0 \implies \lambda = x_A$$

$$\frac{\partial L}{\partial \lambda} = 0 \implies 90P_x + 35 - P_x x_A - y_A = 0$$

Combining λ from the first two conditions

$$x_A = \frac{y_A}{P_x}$$

Substituting into the third condition

$$90P_x + 35 - P_x \frac{y_A}{P_x} - y_A = 0$$

Therefore, the demand for good y by consumer A is

$$y_A = 45P_x + \frac{35}{2}$$

Substituting this into the function of x_A

$$x_A = (45P_x + \frac{35}{2}) \cdot \frac{1}{P_x}$$

Therefore, the demand for good x by consumer A is

$$x_A = 45 + \frac{35}{2P_x}$$

Consumer B's problem:

$$\max_{x_B, y_B} x_B y_B \quad \text{s.t.} \quad P_x x_B + y_B = 30P_x + 25$$

Consumer B's demands:

$$y_B = 15P_x + \frac{25}{2}$$

$$x_B = 15 + \frac{25}{2P_x}$$

Equilibrium:

$$y_A + y_B = 60P_x + 30$$

$$\bar{y}_A + \bar{y}_B = 60.$$

Therefore, the solution of $60P_x + 30 = 60$ will determine the equilibrium price of good x:

$$P_x = \frac{1}{2}$$

Substituting the price into the demands we obtain

$$x_A^* = 80 \quad y_A^* = 40$$

$$x_B^* = 40 \quad y_B^* = 20$$